Getting mathematical concepts and learning proofs by computer aided linear algebra

Wolfgang Fraunholz

University Koblenz – Landau, Germany
w.fraunholz@uni-koblenz.de

Two of the most important aims of learning mathematics are to get the exact mathematical concepts and to learn how to make proofs. To achieve these aims first year students need many examples and exercises as well as visualisations. Especially visualisations were the targets which should be realised by the computer learning environment for linear algebra[1]. The fundamental concepts of linear algebra are vector space, linear dependency of vectors, basis of a vector space, systems of linear equations, matrices, linear mappings and their compositions, scalar product, eigenvalues and eigenvectors.

To get an impression of an n-dimensional vector space one could use examples like this of a storage rack. The needed parts (side frames, shelves, short und long screws) can be listed as vectors with four components. To construct more than one rack one gets the required parts by calculating according the vector space axioms. So we have an exemplar of a four-dimensional vector space. Similar examples can give an impression of higher dimensional vector spaces. With these vectors the concept of linear dependency or linear independency can be shown by calculating, if there are two numbers r and s which fulfill the equation

$$\mathbf{c} = r\,\mathbf{a} + s\,\mathbf{b}\quad[2].$$

Many exercises will ensure the knowledge of this concept. These exercises also include geometrical vectors in two and in three dimensions. The students work interactively with the program.

Learning what will be a basis of a vector space is practised by tasks to

Figure 1

Figure 2

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[1] Lernprogramm Lineare Algebra published by Wolfgang Fraunholz and Max Schröder, Cornelsen Verlag 2003
[2] Bold figures indicate vectors, light figures indicate scalars.

Aufgabe 2 Basis eines Vektorraums

die Vektoren **a**, **b**, **c**, **d**, **e** eine Basis des $\mathbb{R}^5$?

Matrixrechner

$$a = \begin{pmatrix} 9 \\ 6 \\ 9 \\ 7 \\ 3 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ -6 \\ 3 \\ 5 \\ -7 \end{pmatrix} \quad c = \begin{pmatrix} 9 \\ -2 \\ -5 \\ 13 \\ -6 \end{pmatrix} \quad d = \begin{pmatrix} 7 \\ 8 \\ 7 \\ 2 \\ 3 \end{pmatrix} \quad e = \begin{pmatrix} -6 \\ -4 \\ 8 \\ -8 \\ -1 \end{pmatrix}$$

ja

● nein

$$\begin{pmatrix} 9 & 3 & 9 & 7 & -6 \\ 6 & -6 & -2 & 8 & -4 \\ 9 & 3 & -5 & 7 & 8 \\ 7 & 5 & 13 & 2 & -8 \\ 3 & -7 & -6 & 3 & -1 \end{pmatrix}$$

◊ Rang:
⇒ 4

Richtig.

Figure 3

Hilfe

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ -1 & 3 & 0 \end{pmatrix}$$

◊ Rang:
⇒ 3

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

Allgemein ▶
Umformen
Eigenschaften
+/−   Strg+
Matrix   #
Skalarprodukt
Vektorprodukt
Auswerten

Transponieren
Zeilenoperationen ▶
Spaltenoperationen ▶
Gauß-Elimination
Gauß-Jordan
Invertieren

Figure 4

multiplikativ inverse Matrix

Welche Matrix A gehört zur Verkettung f∘g∘h der linearen Abbildungen im Rechenfenster?

Matrixrechner

"Matrix von f" = $\begin{pmatrix} -3 & 1 & -5 \\ 4 & 5 & 4 \\ 3 & 4 & 4 \\ 5 & -1 & 3 \\ -4 & 4 & -5 \end{pmatrix}$, "Matrix von g" = $\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -3 & 3 \end{pmatrix}$, "Matrix von h" = $\begin{pmatrix} 2 & -5 & 1 & -1 \\ 0 & -4 & -1 & -5 \end{pmatrix}$

◊ A = $\begin{pmatrix} 28 & -10 & 29 & 61 \\ 4 & -134 & -29 & -157 \\ -2 & -103 & -28 & -134 \\ -12 & -14 & -17 & -49 \\ 38 & -67 & 26 & 16 \end{pmatrix}$ (Dezimalzeichen ist hier nicht das Komma, sondern der Punkt.)

⇒ "Richtig."

$$\begin{pmatrix} -3 & 1 & -5 \\ 4 & 5 & 4 \\ 3 & 4 & 4 \\ 5 & -1 & 3 \\ -4 & 4 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -5 & 1 & -1 \\ 0 & -4 & -1 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 28 & -10 & 29 & 61 \\ 4 & -134 & -29 & -157 \\ -2 & -103 & -28 & -134 \\ -12 & -14 & -17 & -49 \\ 38 & -67 & 26 & 16 \end{pmatrix}$$

Figure 5

identify tuples of vectors as linear independent and spanning the whole space. If the matrix calculus is already well known this identification can be made by the determination of the rank of the matrix which is build by the vectors. To do so the program has got a matrix calculator whose features should be shown with the following figures.[3]

Commands:
Transformations:
Transpose of a matrix
Operations with rows
Operations with columns
Gauß-Elimination
Gauß-Jordan-Elimination
Revers of a matrix
Properties:
Basis of rows
Basis of columns
Rank of the matrix
Determinant
Characteristic polynom
Eigenvalues
Eigenvectors
Matrix:
Insertion of a matrix with m rows and n columns
Evaluation of a scalar product of vectors
Evaluation of a vector product
For instance one can multiply three matrices, also with inverted matrices written as A^{-1}.

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[3] In the background of this matrix calculator is running the Computer Algebra System MuPAD.

The concept of mapping is introduced by some well known mappings as reflection on straight lines, rotation or dilation. For given points (of a triangle) the students construct by themselves the images and find out the matrix by which the image points can be calculated.

The next figure 7 shows the same task for a dilation. Students always can demand help, so that the task is eased. If they are not able to solve the problem they can choose the solution.

To get the concept of interrelation of mapping matrix and geometrical mapping there is a so called experimental page. Students find a preimage triangle and the image of the triangle generated by the given mapping matrix. As well they find the images of the standard unit vectors. Experimenting by changing the mapping (which can be chosen from a list of mappings; down left) or changing the matrix by giving in real numbers in the positions within the matrix or changing the angle between the images of the unit vectors. The modification of the geometrical mapping is immediatly shown on the screen. Not to be seen is, if the mapping is linear or not. Therefore one has to get the concept of linearity, meaning homology and additivity, by formula

$$f(\lambda\, x) = \lambda\, f(x) \qquad \text{and} \qquad f(x+y) = f(x) + f(y)\,.$$

There are pages where it is explicated by visualisation. The students can directly watch the values of $f(\lambda\, x)$ and $f(x+y)$ of a chosen function which can be varied. They compare the value of $f(\lambda\, x)$ with the value of $\lambda\, f(x)$ and the value of $f(x+y)$ with the value of $f(x) + f(y)$ as it is shown in figure 9. The function $f(x) = x - 1$ is not additive because there are v_1 and v_2 so that $f(v_1+v_2) \neq f(v_1) + f(v_2)$.

Beispiele linearer Abbildungen: Spiegelung an der ersten Winkelhalbierenden

Spiegeln Sie das Dreieck an der 45°-Achse, indem Sie A', B' und C' mit der Maus an die richtigen Stellen ziehen. Was passiert dabei mit den Koordinaten der Punkte?

$$x' = y \qquad y' = x$$

Mit folgendem Ansatz findet man die Matrix $S_{45°}$, die das Spiegelbild berechnet:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Das zugehörige Gleichungssystem:

$$a\cdot x + b\cdot y = y$$
$$c\cdot x + d\cdot y = x$$

liefert die Koeffizienten:

$$a = 0 \;;\; b = 1$$
$$c = 1 \;;\; d = 0$$

Also:

$$S_{45°} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (die Abbildungsmatrix der Spiegelung an der ersten Winkelhalbierenden)
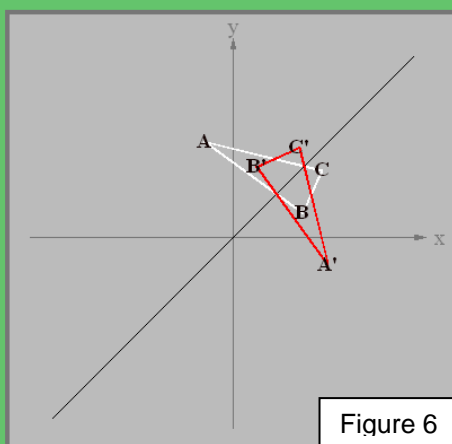
Figure 6

Beispiele linearer Abbildungen: Zentrische Streckung

Strecken Sie das Dreieck zentrisch mit dem Faktor 2, indem Sie A', B' und C' mit der Maus an die richtigen Stellen ziehen.
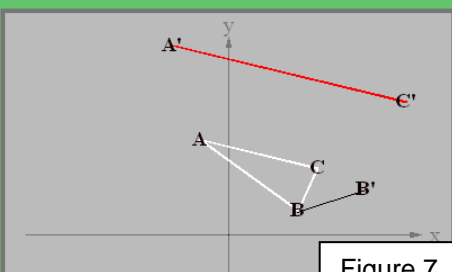
Hilfe   Lösung

Figure 7

Beispiele linearer Abbildungen: Experimentierseite
Stimmt das ganze Bild, wenn die Einheitsvektoren stimmen?
Abbildungsmatrix *(anklickbar)*

$$\begin{bmatrix} \cos(2\cdot78°) & \sin(2\cdot78°) \\ \sin(2\cdot78°) & -\cos(2\cdot78°) \end{bmatrix}$$

Dreieck: $\begin{bmatrix} -0{,}28 \\ 0{,}76 \end{bmatrix} \begin{bmatrix} 0{,}62 \\ 0{,}18 \end{bmatrix} \begin{bmatrix} 0{,}93 \\ 0{,}41 \end{bmatrix}$

Bild-Dreieck: $\begin{bmatrix} 0{,}56 \\ 0{,}58 \end{bmatrix} \begin{bmatrix} -0{,}49 \\ 0{,}42 \end{bmatrix} \begin{bmatrix} -0{,}68 \\ 0{,}75 \end{bmatrix}$

Verändern Sie die Abbildungsmatrix *oder* klicken Sie unten auf eine Abbildung *oder* verschieben Sie e_1' und e_2' mit der Maus.

Spiegelung an der x-Achse
Spiegelung an der y-Achse
Spiegelung an 1. Winkelhalbierenden
Spiegelung an 2. Winkelhalbierenden
Spiegelung an 78°-Ursprungsgerade
Spiegelung am Ursprung

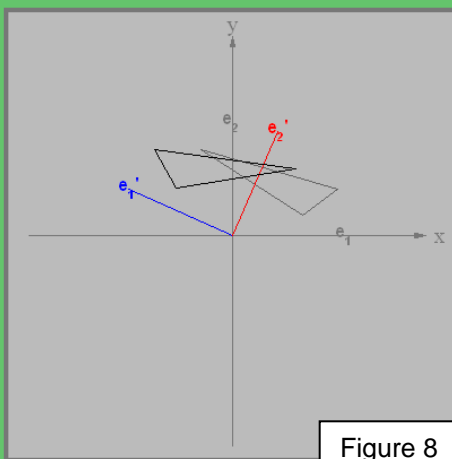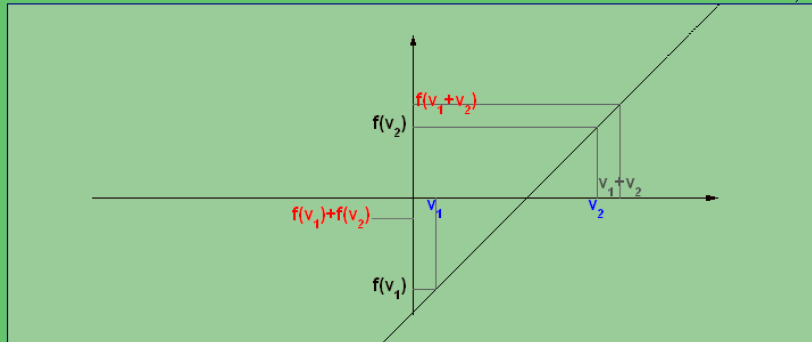Ob die Matrix das Bild immer richtig berechnet, wird in Abschnitt 5.3 mit Hilfe von 5.2 untersucht.

Verstellen Sie den Winkel.

Figure 8

Der Begriff der linearen Abbildung und Eigenschaften
Additivität anschaulich in IR

Prüfen Sie die erste Bedingung der Linearität bei verschiedenen f: IR → IR.
Verschieben Sie v_1 und v_2 mit der Maus!   Verändern Sie f: f(**v**) = v-1   (Beenden Sie Ihre Eingabe mit RETURN.)

f(v_1+v_2)
f(v_2)
v_1+v_2
f(v_1)+f(v_2) v_1 v_2
f(v_1)

$v_1 = \ 0{,}2 \ \rightarrow \ f(v_1) = -0{,}8$
$v_2 = \ 1{,}62 \ \rightarrow \ f(v_2) = \ 0{,}62$
$f(v_1)+f(v_2) = -0{,}18$
$v_1+v_2 = \ 1{,}82 \ \rightarrow \ f(v_1+v_2) = \ 0{,}82$ } bei Linearität gleich für alle v_1, v_2

Figure 9

Der Begriff der linearen Abbildung und Eigenschaften
Additivität anschaulich in der Ebene

Prüfen Sie die erste Bedingung der Linearität bei verschiedenen f: $IR^2 \rightarrow IR^2$.
Verschieben Sie v_1 und v_2 mit der Maus!   f(x | y) = (x | y-1)   (Beenden Sie Ihre Eingabe mit RETURN.)

y
v_1+v_2
v_1
v_2 ○ f(v_1+v_2)
f(v_1)
x
f(v_2)

$v_1 = (\ 0{,}5 \ | \ 1 \ ) \qquad \rightarrow \qquad f(v_1) = (\ 0{,}5 \ | \ 0 \ )$ ○ Vektorpfeile
$v_2 = (\ 1 \ | \ 0{,}5) \qquad \rightarrow \qquad f(v_2) = (\ 1 \ | -0{,}5 \ )$
$f(v_1)+f(v_2) = (\ 1{,}5 \ | -0{,}5 \ )$ } bei Linearität gleich
$v_1+v_2 = (\ 1{,}5 \ | \ 1{,}5 \ ) \rightarrow f(v_1+v_2) = (\ 1{,}5 \ | \ 0{,}5 \ )$ für alle v_1, v_2
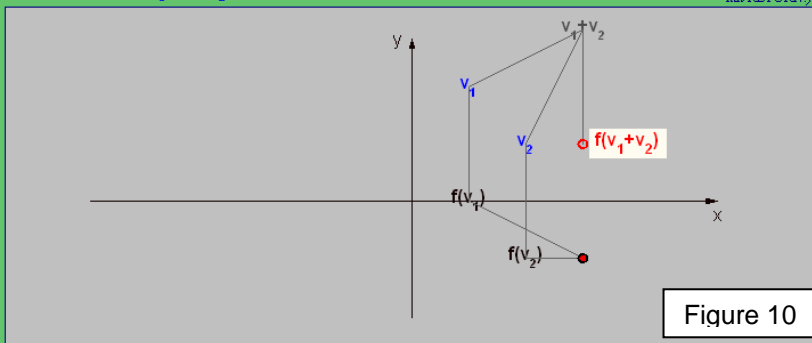
Figure 10

v_1 and v_2 can be moved along the v-axis. The values of v_1, v_2, f(v_1) + f(v_2) and f(v_1+v_2) are shown underneath the graphics. Correspondingly there is a page for functions with two variables. The two variables v_1, v_2 can be moved within the entire plane as it is found in figure 10.

Both figures show non additive functions.

To get the concept of linear mapping also in three dimension an experimental page is provided where the students may create their own three dimensional solids (starting in the plane and moving vertices in the third dimension) or choose a cube, a tetrahedron or an octahedron, on which a mapping is performed.

Figure 11 shows a rotation.

Matrizen als Darstellung linearer Abbildungen – Experimentierseite

Abbildungsmatrix: (in die Liste aufnehmen)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(60°) & -\sin(60°) \\ 0 & \sin(60°) & \cos(50°) \end{pmatrix}$$

Koordinaten	genauere Bildkoordinaten
(−56 | −36 | 92)	(−56 | −97,67433715 | 27,95954
(−93 | 82 | 32)	(−93 | 13,28718708 | 91,58328
(50 | 43 | 197)	(50 | −149,10700455 | 163,868251
(77 | −22 | −9)	(77 | −3,20577137 | −24,83764737
(16 | −34 | 309)	(16 | −284,60184977 | 169,176507

Drehung um 90° um die x-Achse
Drehung um 90° um die y-Achse
Drehung um 90° um die z-Achse
Drehung um 60° um die x-Achse
Drehung um 60° um die y-Achse
Drehung um 60° um die z-Achse

(zurück) (abbilden)   ○ Urbild beim Abbilden lösch
● Bezeichner

C E
y
z x
B
D1 D
B1
A
C1
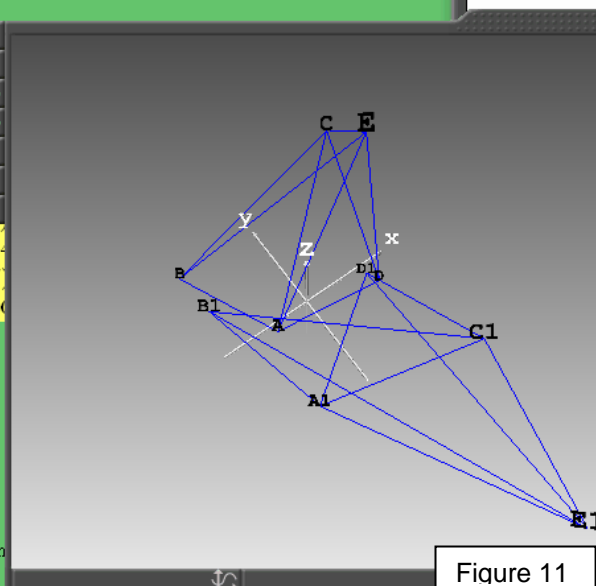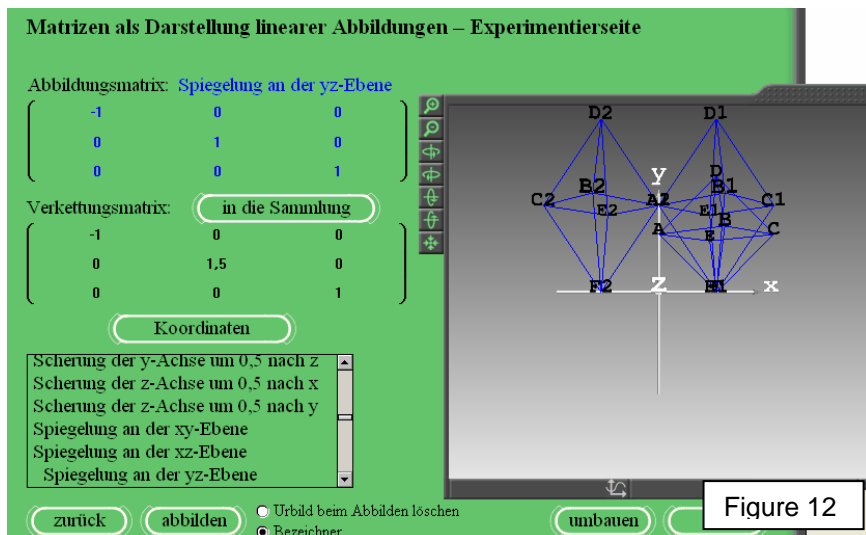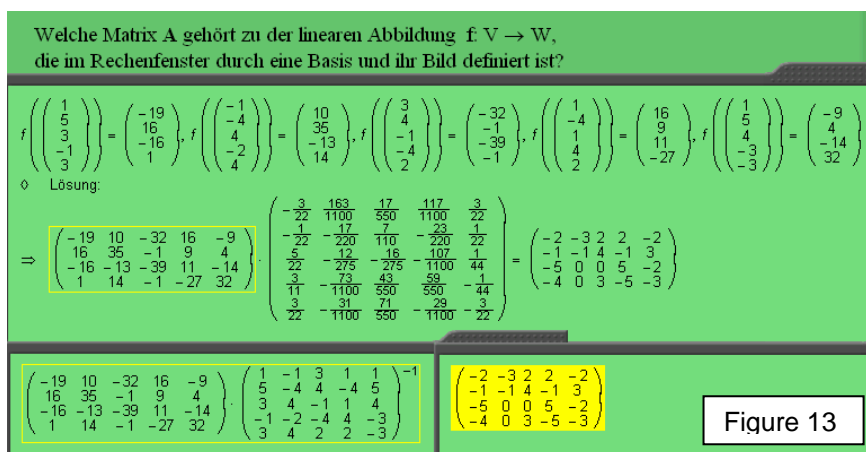A1
E1

Figure 11

Figure 12

Figure 13

Figure 14

The mapping matrix can be changed by selecting one of the mappings in the list or by changing the real numbers within the matrix.

Also the composition of mappings and accordingly the composition of matrices can be tried out with the same experimental page. In figure 12 a reflection of an octahedron at the yz-plane is performed and the composition of two matrices is shown. All variations of mappings or of matrices respectively may be studied. The composition of matrices can easily be acted out using the matrix calculator. This is shown in the lower part of figure 13.
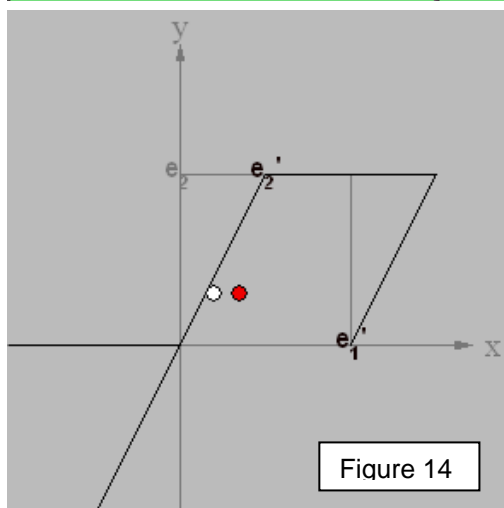
Other important concepts of linear mapping are the concept of the fixed point and the concept of the eigenvector. To learn about fixed points there is a mapping, e. g. $f(\mathbf{v}) = A\,\mathbf{v}$ with $A = \begin{pmatrix} 1 & 0,5 \\ 0 & 1 \end{pmatrix}$. To give an impression of the mapping a square and its image are drawn. The white point is a preimage point, the red point is the image point of the white one. If the white point is moved the red one will follow according the directive of the given mapping. The task is to find the positions where the white point and the red point coincide. These positions are the fixed points. There may be one single fixed point; then of course the origin due to the linearity of the mapping, but also there may be many fixed ponts. Therefor one has to learn to find the fixed points by solving the equation

$$A\,\mathbf{v} = \mathbf{v} \quad \text{or} \quad A\,\mathbf{v} - \mathbf{v} = \mathbf{o} \quad \text{or} \quad (A - E)\,\mathbf{v} = \mathbf{o}.$$

If the arrow from the origin to the white point and as well from the origin to the red point is plotted the students can search for eigenvectors. They have to move the white point round the origin (e. g. on a circumference like in figure 15); the red point will follow according the mapping. So they may find some positions where the (red) image vector is on a straight line with the (white) preimage vector. Vectors in the direction of this straight line are called

eigenvectors and the multiple λ of the length of the two vectors are called eigenvalues. Patiently trying out all possible positions all eigenvalues are found.
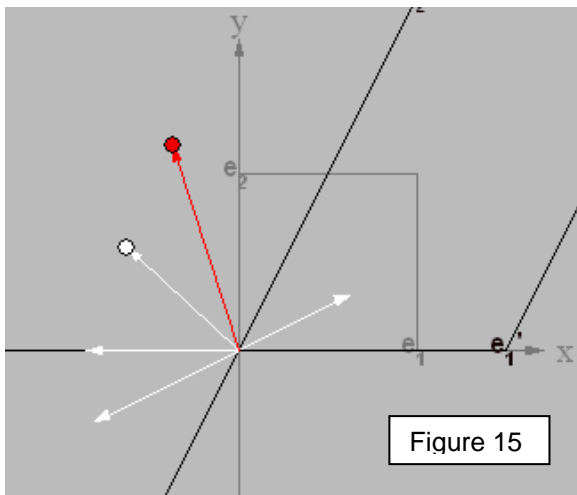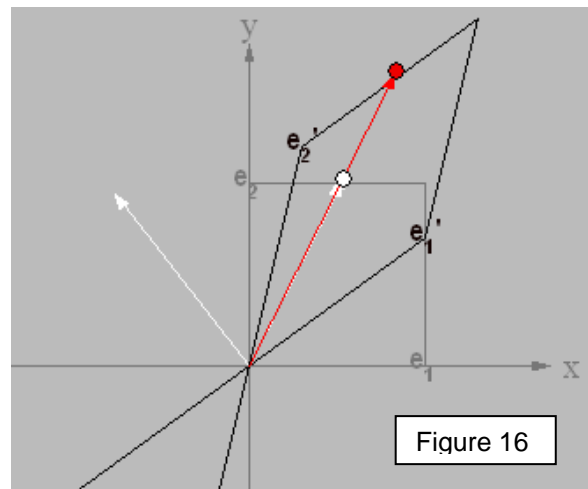

Figure 15

Figure 16

Regrettably this can not be done in three dimensions. In figure 16 one of these positions is demonstrated. The related eigenvalue is inserted on the screen. Students find out that in two dimensions there are mappings without eigenvalues, others with one or with two eigenvalues. The eigenvectors can be figured out by solving the equation

$$A\,\mathbf{v} = \lambda\,\mathbf{v} \quad \text{or} \quad A\,\mathbf{v} - \lambda\,\mathbf{v} = \mathbf{o} \quad \text{or} \quad (A - \lambda\,E)\,\mathbf{v} = \mathbf{o}\,.$$

Figure 17

Non trivial solutions of $(A - \lambda\,E)\,\mathbf{v} = \mathbf{o}$ exist, if the determinant $\det(A - \lambda\,E) = 0$. With this formula one is on the right path to determine eigenvalues of any (n,n)-matrix. With the matrix calculator e. g. the solution for a (3,3)-matrix is like in figure 17.

Learning proofs is a rather complicated task. To get the idea for a proof intuition is needed. This can be learnt by reading proofs and proofs again. Each step within a proof has to be given reasons. Often the steps are signed by the implication sign $\Rightarrow$ without any explication in the written text. The reason is to keep the text short and clearly laid out. The learner however has to be clear in the mind on the grounds of these steps.of deduction.

Figure 18

The computer screen gives the chance to keep the clear layout of the proof and to give all explanations to the essential reasons of the steps of deduction. Blue marked signs like equals signs, less than signs or implication signs will display upon mouse over the needed explanations. Figure 18 shows the example: the proof for the commutativity of vector addition. The equals signs are well founded by the definition of vector addition and by the commutativity of the addition of real numbers.

Figure 19

$$\text{Beh.:} \quad \|x+y\| \leq \|x\| + \|y\|$$

Cauchy-Schwarzsche Ungleichung

$$\text{Bew.:} \quad \|x+y\|^2 = (x+y)\cdot(x+y) = \|x\|^2 + 2x\cdot y + \|y\|^2 \leq \|x\|^2 + 2|x\cdot y| + \|y\|^2 \leq$$

$$\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \Rightarrow \underbrace{\|x+y\| \leq \|x\| + \|y\|}_{\text{Behauptung}}$$

Figure 20

Beweis der Eindeutigkeit von f: $V \to W$ mit $f(v_i) = w_i$

Angenommen, es sei g: $V \to W$ ebenfalls linear mit $g(v_i) = w_i$ (i=1, 2, ..., n).

Da die $v_1, v_2, ..., v_n$ eine Basis von V sind, gibt es zu jedem $x \in V$ Körperelemente $\lambda_1, \lambda_2, ..., \lambda_n$ mit $x = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$.

Wegen der Linearität von f $\quad$ Linearität von f

$$f(x) = f(\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n) = \lambda_1 f(v_1) + \lambda_2 f(v_2) + ... + \lambda_n f(v_n) =$$

$$+ \lambda_2 w_2 + ... + \lambda_n w_n = \lambda_1 g(v_1) + \lambda_2 g(v_2) + ... + \lambda_n g(v_n) =$$

$$= g(\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n) = g(x)$$

The example in figure 19 is the proof of the triangle inequality, where the Cauchy-Schwarz-inequality is used. By mouse over the blue marked signs the declarations are given of the particular steps.

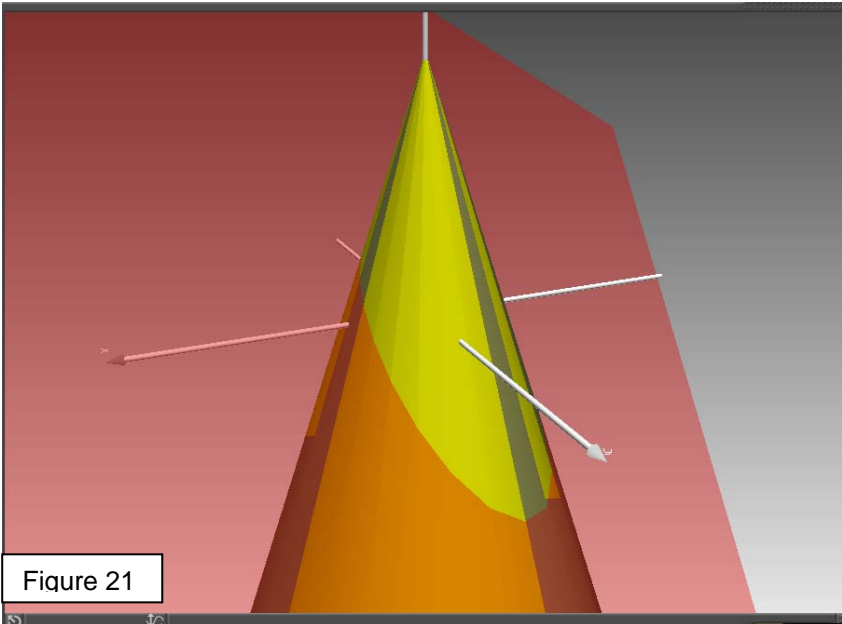The third example (in figure 20) gives the proof for the uniqueness of linear mappings f and g with $f(v_i) = w_i$ and as well $g(v_i) = w_i$, if $v_1, v_2, . . . , v_n$ constitute a basis of the vector space V. The proof uses the linearity of f and of g, which is shown by mouse over the blue equals signs.

To create own visualisations the computer program has got a graphic tool for three dimensional graphics. Students can generate vectors, straight lines, planes, triangels, discs, spheres, cones, cylinders and cuboids (see figure 22). In figure 21 an intersection of a cone with a plane is drawn to get an ellipse. The generated objects can be translated, rotated, linear mapped (see figure 23).

Figure 21

Figure 22

Objekte
Maus-Aktion
Einstellungen

Vektor
Gerade
Ebene
Dreieck
Scheibe
Kugel
Konus
Zylinder
Quader

Figure 23

Objekte
Maus-Aktion
Einstellungen

Schieben
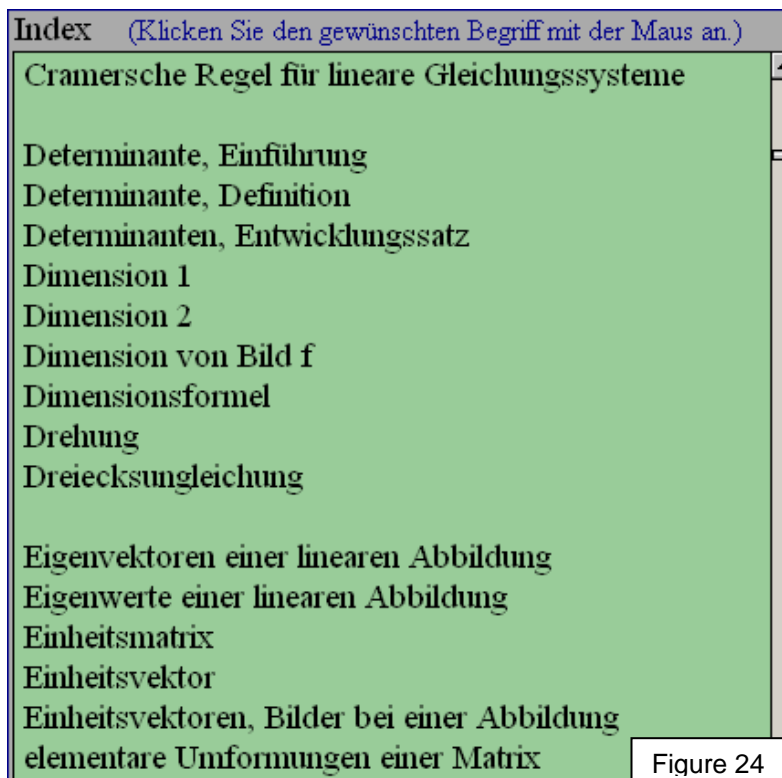Drehen
Linear abbilden
Transparenz ändern
Löschen
nichts

Figure 24

To allow students a fast direct acces to the content and the concepts an index in alphabetical order is provided. As shown in figure 24 for instance they may choose concepts like determinant, dimension, eigenvector and eigenvalue. They need not to leaf through the lessons to get the keyword but they can jump there immediately.

Likewise they can jump to the problems by using the index of tasks (see figure 25). E. g. they find a problem to search out the orthogonal space to a given subspace (see figure 26).
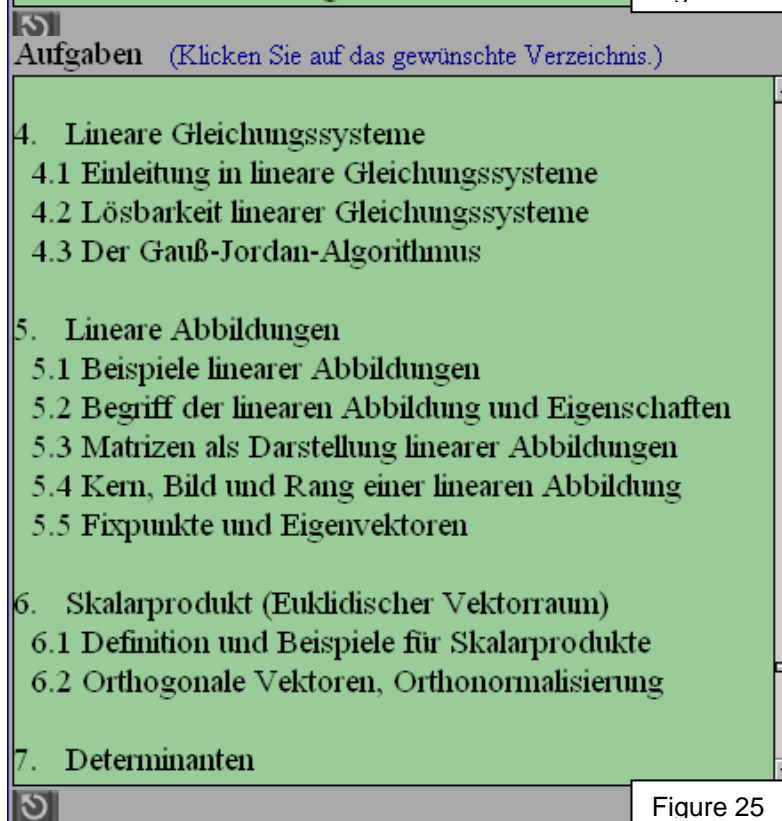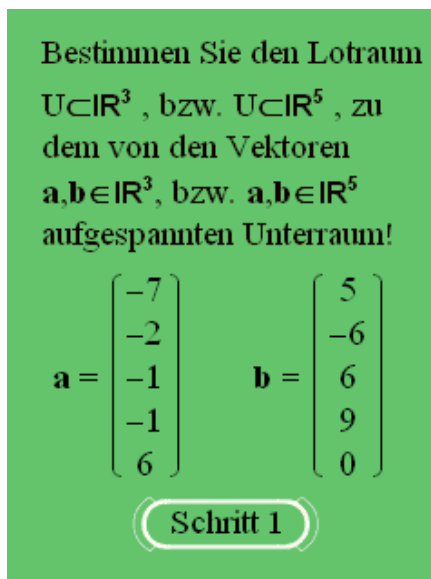
Figure 25

Bestimmen Sie den Lotraum $U \subset \mathbb{R}^3$, bzw. $U \subset \mathbb{R}^5$, zu dem von den Vektoren $a, b \in \mathbb{R}^3$, bzw. $a, b \in \mathbb{R}^5$ aufgespannten Unterraum!

$$a = \begin{pmatrix} -7 \\ -2 \\ -1 \\ -1 \\ 6 \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ -6 \\ 6 \\ 9 \\ 0 \end{pmatrix}$$

Schritt 1

Figure 26