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ANIMATION OF ESSENTIAL CALCULUS CONCEPTS IN MAPLE

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Abstract. The notion of the limit of a real function belongs to essential concepts of calculus. A part of higher schools students has learned the notion in the secondary school mathematics, but, as a rule, the concept of the limit is one of the crucial introductory topics in calculus lectured at universities. Some students understand the concept only intuitively, mainly due to ideas based on the movement of a body. There are different ways how a teacher could demonstrate the correspondence between the intuitive insight and the exact definition of the limit. The paper shows how the concept of limit of a function can be illustrated with Maple animation. The paper also deals with the principle of some other animations, e.g. continuity of function, the derivative of a function at a point, and the Riemann's definite integral. More general problems on teaching and learning are discussed, especially those connected with the special role of learning outcomes in calculus.

Key words. Limit, Continuity, Derivative, Definite Integral, Animation, Computer Algebra Systems, Maple, Learning Outcomes.

Mathematics Subject Classification: Primary 97I40, 97I50; Secondary 97U50.

1 Introduction

The article deals with problems and corresponding procedures how to communicate visually the basic concepts of mathematics in teaching, especially those in calculus (limits of functions and derived notions as continuity, derivative, definite integrals etc.). Here a key role of a teacher is to force learners to touch on analysis of real numbers, even in a dynamic, not static way. On the other hand, it is known that our recipient – a student prefers the information supplied in a visual form. We share the opinion that the access to technological tools affords the ways how to motivate them, how to provide the insight into the core principles, build up the knowledge, and, after acquiring skills, to incorporate them into problems solving. A part of ideas concerning the topic can be found in [4].

2 The limit of a function

The notion of the limit of a real function belongs to essential concepts of calculus. Construction of subsequent calculus concepts uses just the limit concept: the derivative of the function at the point, Riemann's definite integral, multivariable calculus notions, etc. The concept of the limit is also the key notion in iteration processes, especially when searching for equation solutions, or investigation of equilibrium state, of optimal values, for area evaluation, etc. Several mathematical theories include applications of procedures based on limit constructions, and for this reason, they are used fruitfully in other sciences, not only in mathematics.

The intuitive understanding of the notion of limit could be found in considerations of calculus inventors in 17th century. The attempt was connected with the effort of capturing the rules valid for mechanical motion. Issac Newton and Wilhelm Leibniz, both treated now as such inventors, used in argumentations and calculations the notion of the flux, resp. of the infinitely small (infinitesimal) quantity (see [2]). The form of the limit definition has been accepted after 200 years later. It was introduced in lectures of Karl Weierstrass, see, e.g. [5]; he reformulated it, eliminating the notion of infinitesimals. Hence, the key calculus notions required the strong intellectual effort at its very involving – this effort repeats nowaday when a student is forced to move from his/her intuition insight, or from the pre-concept, into the exact interpretation of the concept required.

As it is well-known, searching for the limit of a given real function f of one variable means to ask on values of the function f at those x in its domain that a variable x approaches to the given number a, not being equal to, independently on fact whether a belongs to the domain of the function or not. In case all the values $f(x)$ can be made as close as desired to the number A, a unique one, the number A is to be called the limit of the function f at the point a. Augustin-Louis Cauchy in Preliminaries to *Cours d'Analyse de l'École Royale Polytechnique: Première Partie: Analyse Algébrique (Analysis Course of the Royal École Polytechnique: First Part: Algebraic Analysis) in* 1821 wrote:

When the successively attributed values of one variable indefinitely approach a fixed value in such a way that they finally differ from it by as little as desired, then that fixed value is called the limit of all the others.

Such description has been not fully accepted, as the word „approach" leads to interpretation as a motion or movement is involved and, as we know, mathematics works only with a static concepts. Instead of the intuitive, and dynamic approach, with the support of geometrical interpretation, it turned out that the more appropriate definition of the limit has to be based on the arithmetical approach. It was provided by Bernard Bolzano in 1817, and in a less precise form by Augustin-Louis Cauchy. Karl Weierstrass formalized it in 1851; in his formulation he uses wording „as close as possible"- the variable has been treated as a static one, not depending on time, with any values close to some other given value.

" The limit of the function f at the point a is the real number A, if for each real number ε, $\varepsilon > 0$, there exists a real number δ , $\delta > 0$, such that for all numbers x with $0 < |x - a| < \delta$, it follows $|f(x) - A| < \varepsilon$."

In other words: „The limit of the function f at the point a equals to A, if for any given real number ε, $\varepsilon > 0$, the distance between values $f(x)$ and A is smaller than ε, $\varepsilon > 0$, provided the distance between points x and a is smaller than the given real number δ, $\delta > 0$, and δ depending on the choice of ε."

The work with the preceding definition, having the concrete choice for one pair of numbers ε, δ enables to present one static picture expressing the distances in function values and corresponding arguments of that function. In the classical study, the learner then is forced „to play a game" with those ε, δ, where δ depends on ε. The more intelligent version, supporting the learning (not only learning the limit) in a substantial way, consists in the use of MAPLE tools; then single pictures will move into the animation sequence, moreover, controlled by teacher/learner.

3 Animation in Maple

MAPLE, the Computer Algebra System (CAS, see e. g. [3]) enables to provide numerical computations, and symbolic ones as well. The principle how to generate animation of relations which concern limits with the real number ε, $\varepsilon > 0$, ε as the parameter, consists in the use of procedures. Then the animation itself means to invent the construction of the very fast sequence of appropriate graphs. The MAPLE command recommended for to animate the sequence is animate, see [2]. Working with the task defined above, it turned out that it would be more reasonable to use the command seq: in result of its application, one gets the required sequence of pictures/graphs. Those pictures could be mapped fastly in an ordered sequence using the command display, which is available after calling plots. The command is to be used with the parameter insequence=true.

4 Presentation of essential concepts of calculus

Introducing the notion of the limit of a function and animated it in teaching, based on the previous remarks, one can use therefore also animations derived from that one prepared for the limit as its core.

4.1 The limit of a function at a point

How to interpret geometrically the relation „to be as closed as possible to some given value", the crucial in definition of the limit: drawing, it means that for arbitrary given ε, $\varepsilon > 0$ we construct two paralel lines $y = A + \varepsilon$ and $y = A - \varepsilon$, and we shall find the corresponding δ, $\delta > 0$ such that for any x, $x \neq a$, x between $a - \delta$ and $a + \delta$, it holds $A - \varepsilon < f(x) < A + \varepsilon$, i.e. the graph of the function f is located between horizontal lines $y = A - \varepsilon$, $y = A + \varepsilon$, see [7].

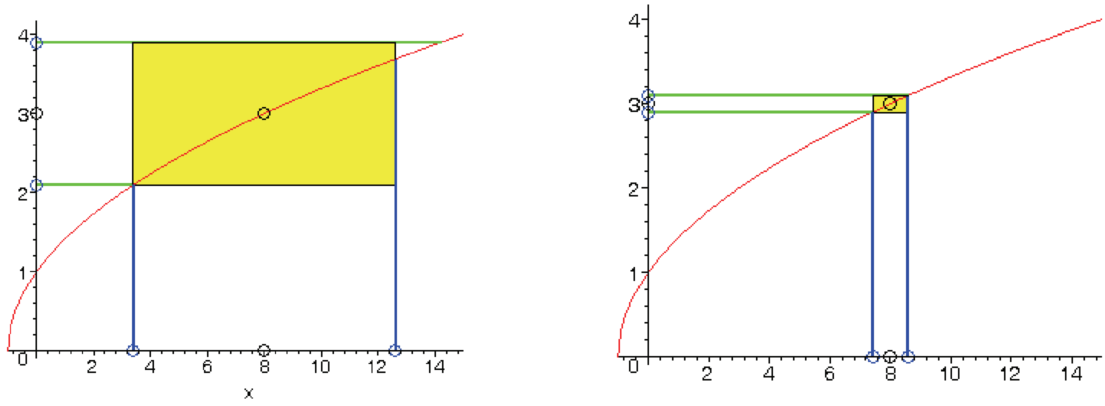

Fig. 1 Geometrical interpretation of the notion of limit of the function $f: y=\sqrt{x-1}$ *at the point a = 8 with the value* $\varepsilon = 9/10$ *or* $\varepsilon = 1/10$.

For the animation sequence, consider the set of new and new, decreasing values of ε, $\varepsilon > 0$ and repeat the construction of the picture of finding the interval between $a-\delta$ and $a+\delta$ as described above. It means, the experiment will help and is worthy to go on. Use the commands available in MAPLE for the experiment – for the construction of such sequence, and in this way animate the notion of the limit of a function. For learning purposes, it is useful to provide details in protocol on parameters values.

4.2 The continuity of a function at a point

MAPLE tools enable also to manipulate with graphs of function in cases when we, in view of the learning purpose, require to join its different parts, fitting some possible parameters, with the aim to get its graph as the graph of the continuous function on a given set. As an example, take the function defined on the set of all real numbers as

$$f: y = \begin{cases} cx^2 + 2x, x < 2, \\ x^3 - cx, x \geq 2, \end{cases}$$

where $c, c \in R$, is the parametr. The problem is formulated as the search for which value of the parameter c the function f should be continuous at the point $x = 2$.

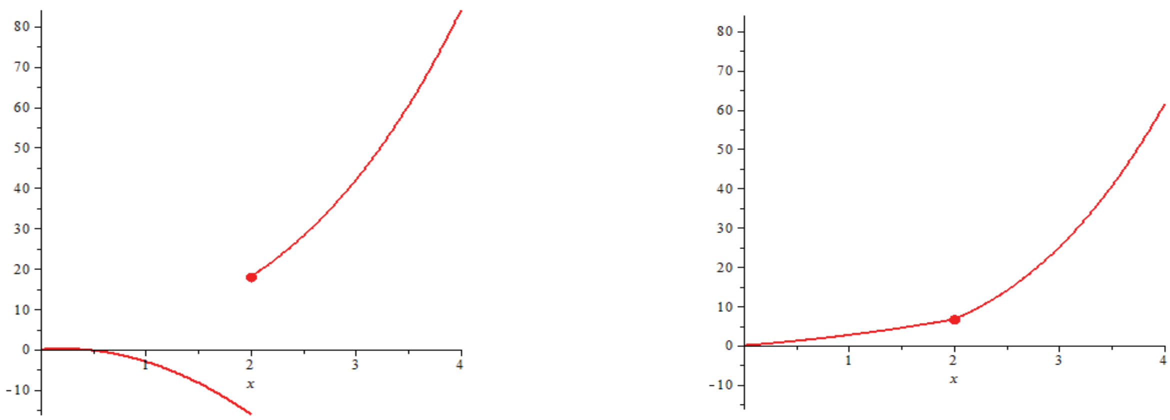

Fig. 2 The function fails to be continuous at $x = 2$ *when the parameter value is* $c = -5$. *The function is continuous at* $x = 2$ *when the parameter value is* $c \doteq 0{,}67$.

4.3 The derivative of a function at a point

The next calculus concept which requires to be acquainted with the concept of the limit is the derivative of the function at a given point. As it is well-known, the geometrical interpretation of this concept means the slope of the tangent line to the graph of the function at the given point. As a support of learning, it would be worthy to follow on the classical approach: to produce the sequence of section lines, each at the given point, and show how section lines are closer and closer to the tangent line at this given point, depending on the existence of the limit for values of section lines slopes. Some phases in running the sequence are shown on Fig. 3. Let us remark that it is valuable to provide also an animation in which we chose as the point in question just the inflection point of the given function.

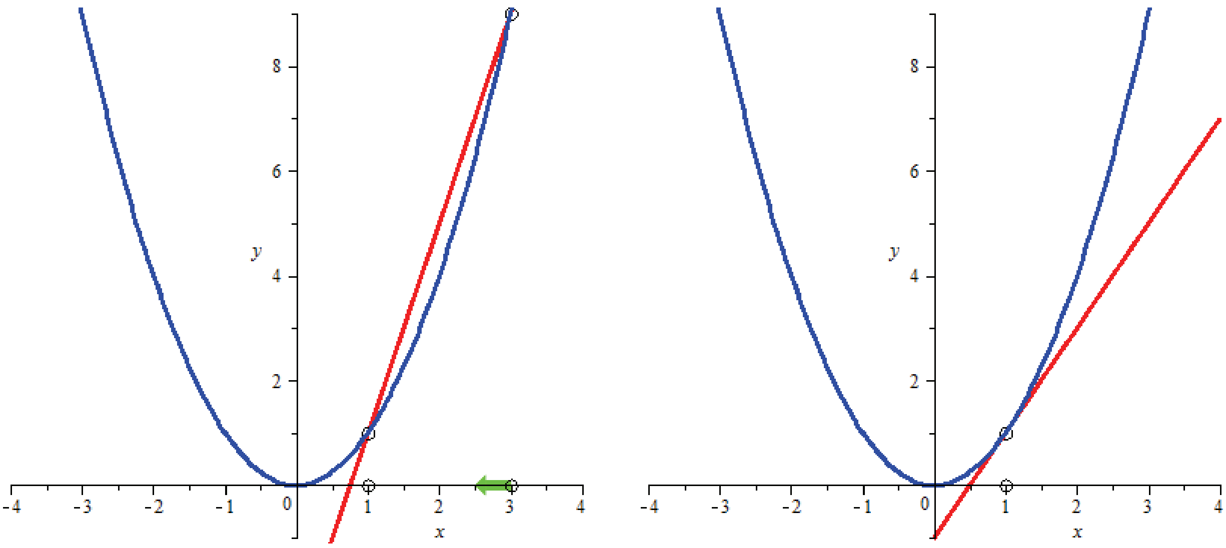

Fig. 3 The geometrical interpretation of the section line – the section line, passing through given points, is approaching to the tangent line at this point. The geometrical interpretation of the derivative of the function at a point: the tangent line, as „the limit location of section lines“, passing through the given point on the graph.

4.4 Riemann's sums

Lecturing on Riemann's definite integral, the animation could be used where for the given bounded function defined on the closed interval and the given division of that interval, one constructs the sequence of Riemann's sums. One item of that sequence is generated as a result of a refinement of preceding interval division. The sequence shows the method how to estimate the area of the plane region under the graph of the function, as the sum of areas of rectangles. The series of single pictures connected into the sequence provide the animation of Riemann's definite integral defined as the limit of the (special) sequence of Riemann's sums. MAPLE suggests to apply the CAS predefined procedure named RiemannSum. Figure 4 shows some sequence items. In teaching, the geometrical interpretation enables to introduce immediately the formal definition of the concept in question, see, e.g. [6]. The definition has been shown, and its principal steps requiring the limit attempt are demonstrated – MAPLE worked as the experiment tool.

5 The value of animation in learning: a wider frame

Why to discuss here the basic, usual mathematical approach in details? Implementation of LMS in learning of mathematical concepts/procedures is included among key activities of the project „REFIMAT" solved at Faculty of informatics and Management, UHK (for its description, see „Acknowledgement" notice added). Faculty students are supplied with the rich base of learning

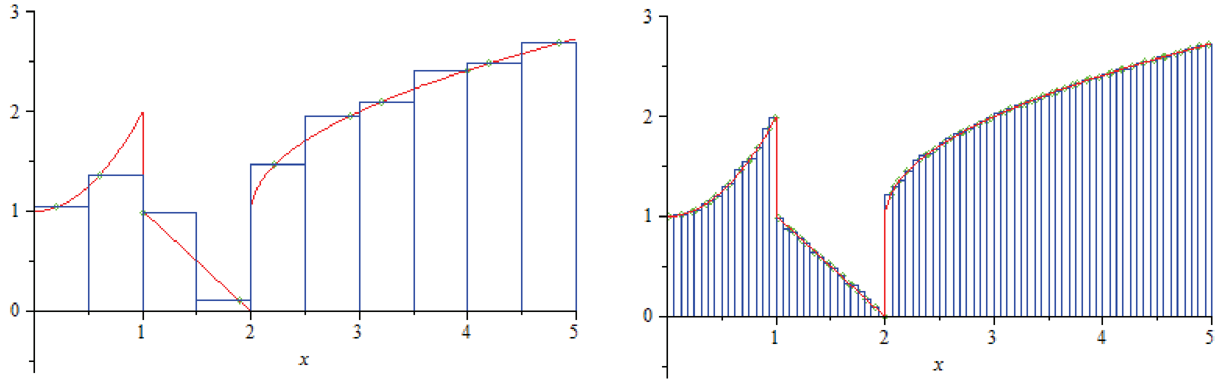

Fig. 4 The construction of Riemann's sums; 10 subintervals of the given interval.
The geometrical interpretation of the definite integral – area of the plane region under the graph of the non-negative function.

material, study supports of different types and styles, including their e-learning versions. Still, research based on the questionnaire in the project, proved the existence of the strong demand of learners on providing them materials of the type „show us how to manage, evaluate, construct, compute, solve – provide us working instructions". Here two streams meet together:

A. Faculty teachers involve learning outcomes into the study of subjects with the mathematical content, as the defined targets for understanding, knowledge and skills. Learning outcomes are defined for programmes or single courses; in the contribution context, learning outcomes of the course mean:

Upon completion of the course, the student will acquire the knowledge, understanding and skills, based on calculus tools, on real functions of one variable, on their basic properties and use in practice, and their roles as principal tools applied for formulating, modeling and solving practice problems.

Concerning the topic limit of a function at the point, learning outcomes for this course module are stated as follows:

Upon completing this section, the student acquires

- *knowledge: using the relevant knowledge on subsets of the set of all real numbers, the students is able to define the limit of the function at a point, he/she provides basic properties of the limit, and he/she learned how to provide the geometrical interpretation of the limit;*
- *he/she understand the notion of the limit at the point, and the local behaviour of the function close to the point, on base of the existence/non-existence of the limit; he understand how to check the existence/non-existence of the limit;*

- *skills: he applies the definition and rules followed from the definition in the numerical evaluation of the limit*

Animations support the learning outcomes in a significant way: they show

- the effective method how to organize a way to an insight, or some kind of experiment on the concepts/procedures,
- they enable, after the starting position, to repeate a great number of necessary steps, which cannot be done by pencil-and-paper method,
- the experiment could be taylored on a learner, on his/her cognitive abilities,
- animations provide steps convergent to the vision, and enrich the vision itself, which cannot be done effectively using only the pencil-and-paper method,
- a learner could be provided by an inspiration how to read/organize an animation, or the experiment, for related topics concepts.

It could be stated therefore that an animation, using CAS included, develops

- understanding assistance
- acquiring of knowledge
- training the skills

Why just MAPLE: instructive system, relatively easily understandable and applicable; also necessarily for the uniform teaching management of large groups of faculty learners, their access to the system is guaranteed and inputs/outputs could be controlled by instructors.

B. The second stream is a valuable declaration of students, showing their active attitude to the study, not to be neglected: „provide us with instructions preferably“; naturally, this does not mean to reduce the study onto the work with solved/unsolved exercises only, or to overestimate the technological aid of the study.

This voice coincides with the common knowledge in the society on how the characteristics of new generations of students is changed; the group of learners born between 1975 - 1981 called X Generation, after 1981 Y Generation (Millennial Generation, Gamer Generation, Net Generation), and some derived notation is used for the next ones etc. show different behaviour patterns, described by Howe and Strauss as follows (see [9]):

- Ability to read visual images—they are intuitive visual communicators
- Visual-spatial skills—perhaps because of their expertise with games they can integrate the virtual and physical
- Inductive discovery—they learn better through discovery than by being told
- Attentional deployment—they are able to shift their attention rapidly from one task to another, and may choose not to pay attention to things that don't interest them
- Fast response time—they are able to respond quickly and expect rapid responses in return

In [9], the characteristics are recollected into a table:

Birth Dates	***1900–1946 Matures***	***1946–1964 Baby Boomers***	***1965–1982 X Generation***	***1982–1991 Net Generation***
Description	Greatest generation	Me generation	Latchkey generation	Millennials
Attributes	Command and control Self-sacrifice	Optimistic Workaholic	Independent Skeptical	Hopeful Determined
Likes	Respect for authority Family Community involvement	Responsibility Work ethic Can-do attitude	Freedom Multitasking Work-life balance	Public activism Latest technology Parents
Dislikes	Waste Technology	Laziness Turning 50	Red tape Hype	Anything slow Negativity

In the next table, Net Gen Characteristics, Learning Principles, Learning Space, and IT Applications are aligned – as the challenge for the necesary management of learning space for individuals raised in the direct contact with ICT:

Net Gen Trait	***Learning Theory Principles***	***Learning Space Application***	***IT Application***
Group activity	Collaborative, cooperative, supportive	Small group work spaces	IM chat; virtual whiteboards; screen sharing
Goal and achievement orientation	Metacognition; formative assessment	Access to tutors, consultants, and faculty in the learning space	Online formative quizzes; e-portfolios
Multitasking	Active	Table space for a variety of tools	Wireless
Experimental; trial and error	Multiple learning paths	Integrated lab facilities	Applications for analysis and research
Heavy reliance on network access	Multiple learning resources	IT highly integrated into all aspects of learning spaces	IT infrastructure that fully supports learning space functions
Pragmatic and inductive	Encourage discovery	Availability of labs, equipment, and access to primary resources	Availability of analysis and presentation applications
Ethnically diverse	Engagement of preconceptions	Accessible facilities	Accessible online resources
Visual	Environmental factors; importance of culture and group aspects of learners	Shared screens (either projector or LCD); availability of printing	Image databases; media editing programs

Interactive	Compelling and challenging material	Workgroup facilitation; access to experts	Variety of resources; no "one size fits all"

Conclusion

Considerations generate also questions of another type, namely on learning space implications (see [9]). The role of a teacher as a facilitator requires to let enough space (in a classroom, or as informal, or a virtual one) for study experiments, searching for learner's own way under own priorities and on the own sequencede steps for a learner. It seems that the intelligent access to technologies together with course management, and with the aim of the effective learning, leads to build up the space as an „Intelligent Tutor"; then, it could be supported by an appropriate institutional LMS, where multiple educational resources are available. Learning science indicates that successful learning is often active, social, and learner-centered ([9]). We share also ideas that with the appropriate use of technology, learning can be made more active, social, and learner centered - but the uses of IT are driven by pedagogy, not technology ([9]).

Animations described in the text are of initial value when approving that Intelligent Tutor as the learning space could be an adequate and functional institutional response to learner's expectations. We hope that this gives the more general meaning to the phenomenon indicated and denoted as *„two streams are meeting together"*.

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References

[1] DUNHAM W. *The Calculus Galery*, Princeton University Press, Princeton, 2005.

[2] EDWARDS C.H. *The Historical Development of the Calculus*, Springer-Verlag, New-York, Berlin, Heidelberg, 1979.

[3] GARVAN F. *The Maple Book*, Chapman & Hall/CRC, 2002.

[4] PRAŽÁK P., GAVALCOVÁ T., PRAŽÁKOVÁ B. Animace pojmu limita funkce v systému počítačové algebry Maple, *Sborník příspěvků ze semináře a soutěže eLearning 2007*, Gaudeamus, Hradec Králové, 2007, pp. 400 – 405. (in Czech)

[5] SCHWABIK Š. Druhá krize matematiky aneb potíže růstu diferenciálního a integrálního počtu, *Sborník Matematika v proměnách věků*, Prometheus, Praha, 1998, pp. 7-60. (in Czech)

[6] SCHWABIK Š. *Integrace v R (Kurzweilova teorie)*, Nakladatelství Karolinum, Praha, 1999. (in Czech)

[7] ŠKRÁŠEK J., TICHÝ Z. *Základy aplikované matematiky I*, SNTL, Praha 1983. (in Czech)

[8] *Augustin-Louis Cauchy and the Rigorisation of Analysis*, [cit. 20.11.2011]. Available on www:

a. http://everything2.com/title/Augustin-Louis+Cauchy+and+the+Rigorisation+of+Analysis
[9] DIANA G. OBLINGER and JAMES L. Oblinger (Editors): *Educating the Net Generation, Educause,* 2005, [cit. 20.11.2011]. Available on www:
a. www.educause.edu/educatingthenetgen/

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